A Theory for Sampling in k-Space - Parallel Imaging as Approximation in a Reproducing Kernel Hilbert Space

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Target Audience: Image reconstruction researchers

Introduction: We show that parallel imaging can be formulated as an approximation of vector-valued functions in a Reproducing Kernel Hilbert Space (RKHS).[1] This formulation provides both a theoretical foundation for reconstruction and sampling in k-space as well as new k-space metrics for interpolation error and noise amplification which go beyond the traditional image-domain g-factor metric.

Theory: All possible multi-channel signals in parallel imaging span only a small subspace of a k-space. It can be shown that this subspace is a RKHS with a matrix-valued kernel derived from the coil sensitivities (Fig. 1).[2] As illustrated in Figure 2, interpolation in k-space from arbitrary, Cartesian or non-Cartesian, samples can then be formulated in the framework of approximation theory.[3] Interpolation weights (cardinal functions), bounds for the interpolation error (power function), and noise amplification (Frobenius norm) can be computed locally for all k-space positions.

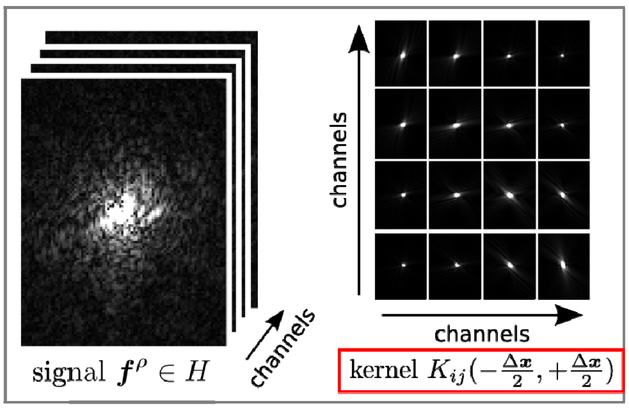

Figure 1: The multi-channel k-space is a Reproducing Kernel Hilbert Space (RKHS) with a matrix-valued kernel. This kernel uniquely characterizes the space.

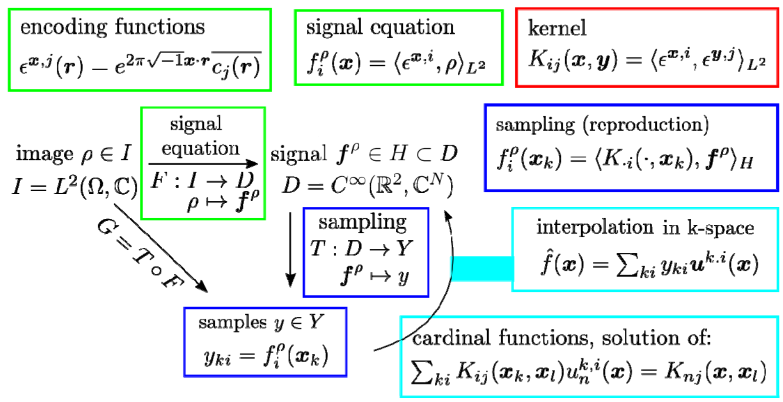

Figure 2 (color): The mathematical relationship between the space of images I, continuous multi-channel k-space signals H (with sensitivities c_j), and discrete samples Y in parallel imaging. Sampling and reconstruction in k-space can be formulated using a RKHS with kernel K.

Methods: Fully-sampled 8-channel data of a human brain (IR-FLASH, TR/TE/TI = 12.2/5.2/450ms, FA = 20º, B_0 = 1.5T) was interpolated to a three-fold oversampled grid by image domain zero-padding. Coil sensitivities were estimated using ESPIRiT[4] and the reproducing kernel of the multi-channel k-space was computed. Several sampling patterns were used for retrospective undersampling (R=4). For all sampling patterns, cardinal and power functions were computed on an oversampled and extended grid.

Results and Discussion: Figure 3 shows reconstructed images, power function and noise amplification in k-space for regular Cartesian, random and Poisson-disc sub-sampling. The power function predicts the interpolation error in k-space and the local Frobenius norm of the cardinal functions predicts noise amplification. Random sampling causes large holes in k-space which lead to high interpolation errors and noise amplification. This is avoided in Cartesian and Poisson-disc sampling, where the later can be used with compressed sensing.

Conclusion: Parallel imaging can be formulated as an approximation in a RKHS, which offers a theoretical framework for image reconstruction in k-space. The derived k-space metrics provide new insights into sampling.

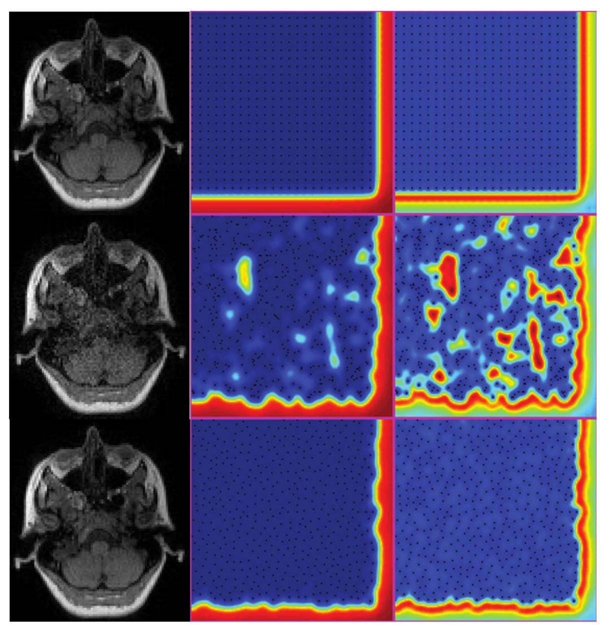

Figure 3 (color): From left to right: Reconstructed images, interpolation error in k-space (power function), and noise amplification in k-space for regular Cartesian (top) random (middle), and Poisson-disc (bottom) sampling patterns (black dots). Random sampling creates holes in k-space which lead to interpolation errors and noise amplification.

References:
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2. Athalye V et al., arXiv:1310.7489 (2013)
3. Wendland H, Scattered Data Approximation, Cambridge University Press (2005)
4. Uecker M et al., Magn Reson Med, 71:990-1001 (2014)